A Note on The Conformal Transformatin of The Step having a Visor of Finite Thickness.

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1. Introduction

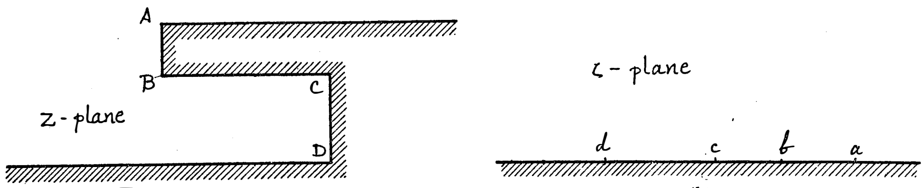
The method of conformal mapping is a very useful technique for obtaining the solution of engineering problems such as hydrobynamics and electrostatics. But, for the difficulties of obtainning the mapping functions the majority of rather complicated problems would not always be able to be solved analytically.

Recently, a useful table of the mapping functions in a number of cases has been published by H.Kober in U.S.A.[1], and by the many substantial examples and their systematic arrangement numerous ploblems will be solved in each field. But as in other cases this fruitful book cannot cover all the problems and the major part of the problems being left unsolved may, of course, be unavoidable matter.

The following is an additional transformation to the above book and it has a wide use especially in the basic problems of the flow around an air intake or exhaust nozzle.

2. Transformation

Consider a step having a visor of finite thickness as shown in Fig.1 (a) and let this plane be Z-plane.



Fg 1 (a) (b)

The conformal transformation of the outer part of the boundary in Z-plane to the above half in C-Plane (cf.Fig.1 (b)) is required. And this transformation is easily constructed by the well known, Schwarz-Christoffel transformation, as follows.

Let the transformed point of A,B,C, and D in the Z-plane be a,b,c and d on the real axis in the ζ-plane, then by the Schwarz-Christoffel theorem

$$\frac{dz}{d\zeta}=k_1\sqrt{\frac{(a-\zeta)\ (b-\zeta)}{(c-\zeta)\ (d-\zeta)}} \qquad \cdots\cdots (1)$$

Or, integrating

$$Z=k_1\int\sqrt{\frac{(a-\zeta)\ (b-\zeta)}{(c-c)\ (d-\zeta)}}\,d\zeta+k_2 \qquad \cdots\cdots (2)$$

where k_1 and k_2 are arbitrary constants. Moreover, k_1 represents the scale effect and k_2 is defined by the positions of the origins of coordinate axis in the respective Gaussian surface. Then, by suitable choise of origins K_2 may become zero.

After a little labourious calculations the integration in the right hand side of eq. (2) is carried out and the whole mapping function becomes as follows:

$$Z = k_1 \int \sqrt{\frac{(a-\zeta)(b-\zeta)}{(c-\zeta)(d-\zeta)}}\, d\zeta = k_1(a-d)(b-d)g \int \frac{cn^2u\, dn^2u}{(1-\alpha^2 sn^2u)^2}\, du$$

$$= k_1(a-d)(b-d)\frac{g}{2\alpha^4}\left\{-\alpha^2 E(u) + (\alpha^2+k^2)u + (\alpha^4-k^2)\Pi(u,\alpha^2) + \frac{\alpha^4 snucnudnu}{1-\alpha^2 sn^2u}\right\} \quad (3)$$

where

$sn^2u = \frac{(a-c)(d-\zeta)}{(a-d)(b-\zeta)}$ etc.: the Jacobian elliptic function,

$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}$: the modulus

$g = \frac{2}{\sqrt{(a-c)(b-d)}}$ $\qquad \alpha^2 = \frac{a-d}{a-c} > 1,$

and

$E(u)$ is a normal elliptic integral of the 2nd kind with modulus k,

$\pi(u,\alpha^2)$ is a normal elliptic integral of the 3rd kind with modulus k.

3. The length of the sides DC, CB and BA on the boundary.

By carrying out the integration on the right hand side of eq. (2) the following relations between the length of the sides and the coordinates on the real axis in ζ-plane are easily obtained.

Namely,

$$\left.\begin{aligned}
\overline{BC} &= k_1\sqrt{(a-c)(b-d)}\left\{E(k') - \frac{b-c}{a-c}K(k') + \frac{(b-c)[(a+b)-(c+d)]}{(a-c)(b-d)}\Pi(\alpha_1^2,k')\right\} \\
\overline{CD} &= k_1\sqrt{(a-c)(b-d)}\left\{E(k) + \frac{a-b}{b-d}K(k) - \frac{(a-b)[(a+b)-(c+d)]}{(a-c)(b-d)}\Pi(\alpha_2^2,k)\right\} \\
\overline{DE} &= k_1\sqrt{(a-c)(b-d)}\left\{E(k') + \frac{a-d}{a-c}K(k') - \frac{(a-d)[(a+b)-(c+d)]}{(a-c)(b-d)}\Pi(\alpha_3^2,k')\right\}
\end{aligned}\right\} \quad (4)$$

where

$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}$: the modulus

$k'^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)} = 1-k^2$: the complementary modulus

K : the complete elliptic integral of the 1st kind

E : the complete elliptic integral of the 2nd kind

Π : the complete elliptic integral of the 3rd kind

with

$$\alpha_1^2 = \frac{c-d}{b-d}, \qquad \alpha_2^2 = \frac{b-c}{a-c}, \qquad \alpha_3^2 = \frac{b-a}{b-d}$$

4. The mapping functions for a step with infinitely thin visor and for a step without visor as the limit case of the general expression (3).

(α) Mapping function for a step having infinitely thin visor.

At the limit end of the tending of b to a, the boundary of Fig.1 (a) becomes a boundary as shown in Fig.2.

After some simple calculations, considering that at the end of limit

$$k^2=\frac{(b-c)(a-d)}{(a-c)(b-d)}=1$$

the limit form of exp.(3) becomes

$$Z=k_1\left(\sqrt{(c-\zeta)(d-\zeta)}+(2a-c-d)\tanh^{-1}\sqrt{\frac{d-\zeta}{c-\zeta}}\right) \quad \cdots\cdots\cdots (5)$$

By the direct application of the Schwarz-Christoffel transformation to this limit boundary we shall be able to obtain the same mappping function without any difficulties.

(β) The mapping function for a step.

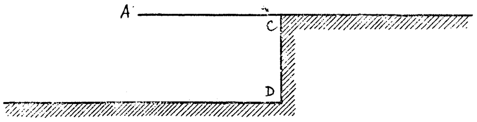
At the limit end of the tending of b to c, the boundary of Fig.1 [2]
becomes a boundary as shown in Fig. 3. And the mapping function becomes as follows,

$$Z=k_1\left(\sqrt{(d-\zeta)(a-\zeta)}+(a-d)\tanh^{-1}\sqrt{\frac{d-\zeta}{a-\zeta}}\right) \quad \cdots\cdots\cdots (6)$$

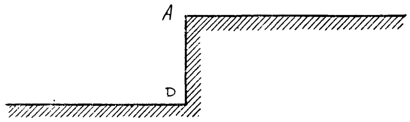
by considering that at this limit

$$k^2=\frac{(b-c)(a-d)}{(a-c)(b-d)}=0$$

and also this same mappping function will be obtained by the direct application of the Schwarz-Christoffel transformation to the same boundary.(2)



Fg. 2



Fg. 3

5. Concluding remarks.

As shown above the mappping functions have been found for those very special cases. However, the appearance of the incomplete ellptic integral of the 3rd kind and the lack of the complete table of this integral makes difficult to take a step forward. Hence the application of the transformation of eq. (3) to the physical and engineering problems is impossible without tedious numerical calculations.

As an approximate solution of the flow around the two dimensional air intake, the uniform flows around the limit boundary of Fig.2 with suctions distributed on CD and the same boundary without any suctinos have been calculated, and the results are in printing.

References.

(1) Kober,H. : Dictionary of Coformal Reresentations, ,Dover, 1952.

(2) loc.cit. p. 162.

(3) Byrd,P.F., and M.D.Fredman : Handbook of Elliptic Integrals for Engineers and physicists, Springer, 1954.